Two decreasing measures for simply typed $\lambda-$terms

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Outline

- Some Strong Normalization proofs
- Decreasing measures
- The auxiliary (non-erasing) λ^m calculus
- The $\mathcal{W}$ measure: based on operations over λ^m
- The $\mathcal{T}^m$ measure: extends a Turing's measure

SN proofs: a brief overview

Reducibility candidates [Tait, Girard]

Most widely known and used | Succint | Great adaptability

Redex degrees + λI

Turing's measure
implies
Weak Normalization

Translation to λI (non-erasing) + WN
allows for
Strong Normalization

Increasing functionals [Gandy, de Vrijer]

1. Map types to sets of strictly increasing functions ($\mathcal{IF}$)

$$\mathcal{IF}_\tau = \mathbb{N} \qquad \mathcal{IF}_{A \to B} = \mathcal{IF}_A \Rightarrow \mathcal{IF}_B$$

2. Map terms to $\mathcal{IF}$ $\quad [\cdot] : A \to \mathcal{IF}_A$
3. Project $\mathcal{IF}$ to $\mathbb{N}$ $\quad \star : \mathcal{IF}_A \to \mathbb{N}$
4. $\star$ constitutes a **decreasing measure** for terms

Decreasing measures

Definition (Decreasing measure)

$$\# : \Lambda \to \mathit{WFO} \qquad M \to_\beta N \implies \#(M) > \#(N)$$

A decreasing measure implies SN

Why to define a decreasing measure?

- Provides more information
- Allows for deeper analysis (*e.g.* remaining steps)

An ideal measure

- simple definition
- easy to compute
- simple *WFO*
- easy to prove

Our work

- Two measures, $\mathcal{W}$ and $\mathcal{T}^m$
- Neither are ideal
- Contribution to better understanding why simply typed $\lambda-$terms terminate

Turing's measure

Definition

Redex degrees

height of a type
e.g. $\mathsf{h}((\tau \to \tau) \to (\tau \to \tau)) = 2$

degree of a redex
e.g. $\delta((\lambda x.x^{\tau\to\tau})t) = 2$

Example

$$I = \lambda x^{\tau}.x \qquad \delta(I\,x) = \mathsf{h}(\tau \to \tau) = 1$$

$$K = \lambda x^{\tau}.\lambda y^{\tau}.x \qquad \delta(K\,(I\,x)) = \mathsf{h}(\tau \to \tau \to \tau) = 2$$

$$\underbrace{K\,(\underbrace{I\,x}_{1})}_{2}\,(\underbrace{I\,x}_{1})$$

Idea: multiset of the redex degrees of M

$$\mathcal{T}(M) = [\ d \mid R \text{ is a redex of degree } d \text{ in } M\ ] \qquad [2, 1, 1]$$

Turing's measure

Weak normalization

Redex creation [Lévy, 1978]

identity applied to a λ	$(\lambda x.x)\,(\lambda y.M)\,N$	$\to_\beta$	$(\lambda y.M)\,N$
λ body is a λ	$(\lambda x.\lambda y.M)\,N\,O$	$\to_\beta$	$(\lambda y.M[N/x])\,O$
replaced var in app position	$(\lambda x.\ldots x\,M\ldots)\,(\lambda y.N)$	$\to_\beta$	$\ldots(\lambda y.N)\,M[\lambda y.N/x]\ldots$

Two crucial observations [Turing, 1940s]

- a redex cannot create redexes of greater or equal degree
- a redex can copy redexes of any degree

Choosing the redex to contract

- has the greatest degree
  - less occurrences of greater element
  - avoid copying redexes of greater degree
- rightmost occurrence of that degree
  - avoid copying redex of the same degree

Example

$$\mathcal{T}(\underbrace{K\,(\underbrace{I\,x}_{\text{S1}})}_{\text{R2}}\,(\underbrace{I\,x}_{\text{T1}})) = [\underset{\text{R}}{2}, \underset{\text{S}}{1}, \underset{\text{T}}{1}]$$

$$\mathcal{T}(\underbrace{(\lambda y.\underbrace{I\,x}_{\text{S1}})\,(\underbrace{I\,x}_{\text{T1}})}_{\text{U1}}) = [\underset{\text{S}}{1}, \underset{\text{T}}{1}, \underset{\text{U}}{1}]$$

The auxiliar λ^m-calculus

Motivation: we can define a decreasing measure from an increasing measure, WCR and WN

Definition

$$t \;::=\; x \mid \lambda x.t \mid t\,t \mid t\{t\} \qquad (\lambda x.t)s \to_m t[s/x]\{s\} \qquad t\,\underbrace{\{s\{r\}\}\{u\}}_{L} \implies t\mathsf{L}$$

weight of a term: amount of memories $\quad \mathsf{w}(x\{y\{z\}\}\{w\}) = 3$

Lemma

1. *λ^m satisfies subject reduction*
2. *λ^m is confluent*

Simplification

- $\mathsf{S}_D(t)$: simultaneous contraction of D redexes
- $\mathsf{S}_*(t)$: iterative $\mathsf{S}_i(t)$ $\quad \mathsf{S}_1(\ldots \mathsf{S}_D(t)\ldots)$ $\quad (D \text{ max } \delta)$

Lemma

3. $t \to_m^* \mathsf{S}_*(t)$
4. $\mathsf{S}_*(t)$ *normal form of* t

Forgetful reduction

$t\{s\} \triangleright t \qquad e.g.\ It \to_m t\{t\} \triangleright t$

Lemma

5. *$\triangleright$ commutes with $\to_m$*
6. *$M \to_\beta N$ implies $M \to_m s \triangleright N$*

The first measure: counting memories

The $\mathcal{W}$ measure

λ^m is **increasing**: $\mathsf{w}(t)$

$$(\lambda x.t)\mathsf{L}s \to_m t[s/x]\{s\}\mathsf{L}$$

Idea: normal form of M has more memories than the normal form of N

Definition

$$\mathcal{W}(M) = \mathsf{w}(\mathsf{S}_*(M))$$

$$M \xrightarrow{\beta} N$$

$$\mathsf{S}_*(M) \quad \triangleright \quad \mathsf{S}_*(N)$$

$$\mathsf{w}(\mathsf{S}_*(M)) > \mathsf{w}(\mathsf{S}_*(N))$$

A remark: it is not necessary to have a proof of WN to define $\mathcal{W}$

Theorem

$$M \to_\beta N \qquad \Longrightarrow \qquad \mathcal{W}(M) > \mathcal{W}(N)$$

The second measure: extending Turing's one

Turing's measure

What about contracting *any* redex?

Recall that: a redex can copy redexes of greater or equal degree

For instance

- $M \underset{R}{\rightarrow_\beta} N$
- R with $\delta(R) = 1$ copies a redex S with $\delta(S) = 2$

$$\mathcal{T}(M) = [\underset{\mathsf{S}}{2}, \underset{\mathsf{R}}{1}] \qquad \mathcal{T}(N) = [\underset{\mathsf{S'}}{2}, \underset{\mathsf{S''}}{2}]$$

Our proposal: to adapt the measure so that it decreases by contracting *any* redex

A first attempt: $\mathcal{T}'$ measure

Problems

$(>)$ A redex copies redexes of greater degree $\mathcal{T}(M) = [2,1] \longrightarrow \mathcal{T}(N) = [2,2]$

$(=)$ A redex copies redexes of same degree $\mathcal{T}(M) = [1,1] \longrightarrow \mathcal{T}(N) = [1,1]$

Idea

$i)$ generalize $\mathcal{T}$ to a family indexed by degrees, so *e.g.*

$$\mathcal{T}'_2(M) = [\underset{\mathsf{S}}{2}, \underset{\mathsf{R}}{1}] \quad \text{and} \quad \mathcal{T}'_1(M) = [\underset{\mathsf{R}}{1}]$$

$ii)$ instead of counting redex degrees in an isolated way, consider also the information about remaining smaller redexes, so *e.g.*

$$\mathcal{T}'_2(M) = [\ (\underset{\mathsf{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\mathsf{R}}{1}, [])\ ] \qquad \mathcal{T}'_1(M) = [\ (\underset{\mathsf{R}}{1}, [])\ ]$$

Definition

- $\mathcal{T}'_D(M) = [(i, \mathcal{T}'_{i-1}(M)) \mid R \text{ is a redex of degree } i \leq D \text{ in } M]$
- $\mathcal{T}'(M) = \mathcal{T}'_D(M)$ where D is the maximum degree of M

A second attempt: $\mathcal{T}^\beta$ measure

Definition (Development of a set of redexes)

Reduction sequence where each step corresponds to a **residual** of a redex **in the set**

- A **residual** is a copy of a redex left after contracting another
- Notation: $\rho : M \xrightarrow{D}{}^{*}_{\beta} M'$

Idea

i) generalize $\mathcal{T}$ to a family indexed by degrees $\mathcal{T}^\beta_D$

ii) instead of isolatedly counting redexes degrees, consider:

- from set of redexes of degree D
- target M' from every development $\rho : M \xrightarrow{D}{}^{*}_{\beta} M'$
- multiset of those $\mathcal{T}^\beta_{D-1}(M')$

Definition

$$\mathcal{T}^\beta_D(M) = [\ (i, \mathcal{V}^\beta_i(M)) \mid R \text{ is a redex of degree } i \leq D \text{ in } M\ ]$$

$$\mathcal{V}^\beta_D(M) = [\ \mathcal{T}^\beta_{D-1}(M') \mid \rho : M \xrightarrow{D}{}^{*}_{\beta} M'\ ]$$

Problem: our technique to prove it decreases does not work because of erasing

$\mathcal{T}^m$ measure

Idea

i) generalize $\mathcal{T}$ to a family indexed by degrees $\mathcal{T}^m_D$

ii) instead of isolatedly counting redexes degrees,
consider the multiset of the measures $\mathcal{T}^m_{D-1}$ of every target of a development of degree D

Definition

$$\mathcal{T}^m_D(t) = [\ (i, \mathcal{V}^m_i(t)) \mid R \text{ is a redex of degree } i \leq D \text{ in } t\ ]$$

$$\mathcal{V}^m_D(t) = [\ \mathcal{T}^m_{D-1}(t') \mid \rho : t \xrightarrow{D}{}^{*}_{m} t'\ ]$$

Lemmas

- **Forget/decrease**: forgetful reduction $\triangleright$ decreases $\mathcal{T}^m$
- **High/increase**: contracting a redex of degree $D > i$ increases (non-strictly) $\mathcal{T}^m_i$
  only $\leq i$, no D, in $\mathcal{T}^m_i$ — no erasing of any $\leq i$ — maybe copies of $\leq i$
- **Low/decrease**: contracting a redex of degree $i < D$ decreases (strictly) $\mathcal{T}^m_D$
  injective mappings from devs of $\mathcal{V}^m_D(N)$ to devs of $\mathcal{V}^m_D(M)$

Theorem

$$M \to_\beta N \qquad \Longrightarrow \qquad \mathcal{T}^m(M) > \mathcal{T}^m(N)$$

Conclusion and future work

Conclusion

- Some Strong Normalization proofs
- Decreasing measures
- Auxiliar non-erasing λ^m calculus
- $\mathcal{W}$ measure: based on weight (or accumulated memory) of terms in λ^m
- $\mathcal{T}^m$ measure: based on anidated multisets of measures of target developments

Future work

- Extend these measures to System F
- Formalize them in a proof assistant

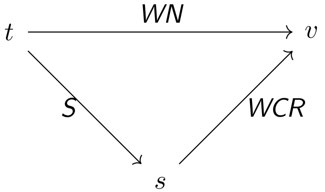
The auxiliar λ^m−calculus

Motivation

β is erasing $\qquad (\lambda x.y)t \to_\beta y$

A motivation not to erase

- Klop-Nederpelt lemma $\quad \mathit{INC} \wedge \mathit{WCR} \wedge \mathit{WN} \implies \mathit{SN} \wedge \mathit{CR}$
- We can obtain a decreasing measure from $\mathit{INC} \wedge \mathit{WCR} \wedge \mathit{WN}$
  - by WN there is a normal form v for any t
  - by WCR it is the same for every reduct s of t
  - by INC $\mathit{inc}(t) < \mathit{inc}(s) < \mathit{inc}(v)$
  - $\mathit{dec}(t) = \mathit{inc}(v) - \mathit{inc}(t)$



Turing's measure "failing" example

Example: copying a redex of greater degree

$$
\begin{aligned}
I_1 &= \lambda x^{\tau}.x & \delta(I_1\,x) &= \mathsf{h}(\tau \to \tau) &&= 1 \\
I_2 &= \lambda x^{\tau\to\tau}.x & \delta(I_2\,I_1) &= \mathsf{h}((\tau \to \tau) \to (\tau \to \tau)) &&= 2 \\
K &= \lambda x^{\tau}.\lambda y^{\tau}.x & \delta(K\,\_) &= \mathsf{h}(\tau \to \tau \to \tau) &&= 2 \\
S_{K\,I} &= \lambda x^{\tau}.K\,x\,(I_1\,x) & \delta(S_{K\,I}\,\_) &= \mathsf{h}(\tau \to \tau) &&= 1
\end{aligned}
$$

$$
\mathcal{T}(\underbrace{S_{\underbrace{K}_{\mathsf{S2}}\,\underbrace{I}_{\mathsf{T1}}}\,(\underbrace{I_2\,I_1}_{\mathsf{U2}}\,x))}_{\mathsf{R1}} = \{\underset{\mathsf{S}}{2}, \underset{\mathsf{U}}{2}, \underset{\mathsf{R}}{1}, \underset{\mathsf{T}}{1}\}
\qquad
\mathcal{T}(K\,\underbrace{(\underbrace{I_2\,I_1}_{\mathsf{U'2}}\,x)}_{\mathsf{S2}}\,\underbrace{(I_1\,(\underbrace{I_2\,I_1}_{\mathsf{U'2}}\,x))}_{\mathsf{T1}}) = \{\underset{\mathsf{S}}{2}, \underset{\mathsf{U'}}{2}, \underset{\mathsf{U'}}{2}, \underset{\mathsf{T}}{1}\}
$$

A first attempt: $\mathcal{T}'$ measure

A working? example $(>)$

Definition

- $\mathcal{T}'_D(M) = [(d, \mathcal{T}'_{d-1}(M)) \mid R$ is a redex of degree $d \le D$ in $M]$
- $\mathcal{T}'(M) = \mathcal{T}'_D(M)$ where D is the maximum degree of M

Example

$$M = \underbrace{S_{\underbrace{K}_{\text{S2}}\ \underbrace{I}_{\text{T1}}}\ (\underbrace{I_2\, I_1}_{\text{U2}}\, x)}_{\text{R1}} \longrightarrow_\beta \underbrace{K\,(\underbrace{I_2\, I_1}_{\text{U}'2}\, x)}_{\text{S2}}\ \underbrace{(I_1\,(\underbrace{I_2\, I_1}_{\text{U}''2}\, x))}_{\text{T1}} = N$$

$$\mathcal{T}'_2(M) = [\ (\underset{\text{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{U}}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, [])\ ] \qquad \mathcal{T}'_1(M) = [\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, [])\ ]$$

$$\mathcal{T}'_2(N) = [\ (\underset{\text{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{U}'}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{U}''}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{T}}{1}, [])\ ] \qquad \mathcal{T}'_1(N) = [\ (\underset{\text{T}}{1}, [])\ ]$$

$$(2, [\ (1, []),\ \ (1, [])\ ]) \quad > \quad (2, [\ (1, [])\ ])$$

A first attempt: $\mathcal{T}'$ measure

A failing example $(=)$

Definition

- $\mathcal{T}'_D(M) = [(d, \mathcal{T}'_{d-1}(M)) \mid R$ is a redex of degree $d \leq D$ in $M]$
- $\mathcal{T}'(M) = \mathcal{T}'_D(M)$ where D is the maximum degree of M

Example Example

$$M = \underbrace{\underbrace{S_{\underbrace{K}_{S2}\ \underbrace{I}_{T1}}}\ \underbrace{(I_1\ x)}_{U1}}_{R1} \longrightarrow_\beta \underbrace{K\ (\underbrace{I_1\ x}_{U'1})}_{S2}\ \underbrace{((\underbrace{I_1\ x}_{U''1}))}_{T1} = N$$

$$\mathcal{T}'_2(M) = [\ (\underset{S}{2}, \mathcal{T}'_1(M)),\ (\underset{R}{1}, []),\ (\underset{T}{1}, []),\ (\underset{U}{1}, []),\ ] \qquad \mathcal{T}'_1(M) = [\ (\underset{R}{1}, []),\ (\underset{T}{1}, []),\ (\underset{U}{1}, []),\ ]$$

$$\mathcal{T}'_2(N) = [\ (\underset{S}{2}, \mathcal{T}'_1(M)),\ (\underset{T}{1}, []),\ (\underset{U'}{1}, []),\ (\underset{U''}{1}, [])\ ] \qquad \mathcal{T}'_1(N) = [\ (\underset{T}{1}, []),\ (\underset{U'}{1}, []),\ (\underset{U''}{1}, [])\ ]$$

$$(2, [\ (1, []),\ (1, []),\ (1, [])\ ]) \quad = \quad (2, [\ (1, []),\ (1, []),\ (1, [])\ ])$$

A second attempt: $\mathcal{T}^\beta$ measure

Definition

$$\mathcal{T}^\beta_D(M) = [\ (i, \mathcal{V}^\beta_i(M)) \mid R \text{ is a redex of degree } i \leq D \text{ in } M\ ]$$

$$\mathcal{V}^\beta_D(M) = [\ \mathcal{T}^\beta_{D-1}(M') \mid \rho : M \xrightarrow{D}{}^{*}_{\beta} M'\ ]$$

Reasoning about the auxiliar measure $\mathcal{V}^\beta_D$

Consider

$$\underset{R}{M \to_\beta N} \qquad \mathcal{T}^\beta_D(M) > \mathcal{T}^\beta_D(N) \qquad \mathcal{V}^\beta_D(M) > \mathcal{V}^\beta_D(N)$$

1. Copying a redex of same degree $(=)$
   - injective mapping from devs of $\mathcal{V}^m_D(N)$ to devs of $\mathcal{V}^m_D(M)$ $\qquad R\rho : M \to_\beta N \to^*_\beta N'$

$$\mathcal{V}^\beta_D(M) > \mathcal{V}^\beta_D(N) \qquad \mathcal{T}^\beta_D(M) > \mathcal{T}^\beta_D(N)$$

2. Copying a redex of higher degree $(>)$
   - not clear the same can be done: a ρ may erase R

$$\mathcal{V}^\beta_D(M') = \mathcal{V}^\beta_D(N') \qquad \mathcal{T}^\beta_D(M') = \mathcal{T}^\beta_D(N')$$